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1990 J. Phys. A: Math. Gen. 23 L291

(http://iopscience.iop.org/0305-4470/23/7/001)

J. Phys. A: Math. Gen. 23 (1990) L291-L296. Printed in the UK

LETTER TO THE EDITOR

On the polar decomposition of the quantum SU(2) algebra

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Received 1 February 1990

Abstract. Polar decomposition of the generators of the $\mathrm{SU}(2)_q$ algebra is performed and its contraction limit to the deformed oscillator algebra is considered. Using the phase operator occurring in the decomposition and its conjugate, we write a realisation of the $\mathrm{SU}(2j+1)$ algebra into which we embed the deformed SU(2) realisation. Finally we discuss the classical limit of the polar decomposition and the case of the quantum SU(2) dynamics.

Quantum groups have their origin in the quantum inverse problem method [1] and the first such structure, i.e. $\mathrm{SL}(2)_q$, has appeared in studies of the Yang-Baxter equation [2]. Subsequent developments have shown that a Hopf algebra description of quantum groups is the appropriate one [3, 4]. Also their relation to non-commutative geometry and the theory of knots and links has attracted great interest. In physics, quantum groups are related to the solutions of integrable systems, to certain problems in statistical physics and to conformal field theories. Also an extension of the theory of quantum groups to supersymmetric quantum Lie groups has been achieved [5, 6].

On the other hand the problem of polar decomposition of the a, $a^{\dagger}$ operators of the harmonic oscillator goes back to Dirac [7] who tried to define the Hermitian phase operator as the quantum analogue of the classical phase variable. Progress in this problem has been reviewed in [8]. Recent developments on the problem [9] have shown that the Hermitian phase operator of the harmonic oscillator and the polar decomposition of a, $a^{\dagger}$, although not defined in the whole Fock space, can be defined in a finite subspace with dimension s and the physically interesting quantities such as expectation values and transition probabilities, are first evaluated in the s-dimensional subspace and then by taking the limit $s \to \infty$.

In this letter we consider the quantum algebra $\mathrm{SU}(2)_q$, for which we perform a polar decomposition. Using the technique of group contraction we derive the q-commutator for the deformed a, $a^{\dagger}$ and their contracted polar decompositions. Also we give the embedding of the matrices of the $\mathrm{SU}(2)_q$ generators in the algebra $\mathrm{SU}(2j+1)$ the generators of which are given in terms of the phase operator of the polar decomposition h, and its conjugate g. Finally, motivated by the quantum plane condition which is satisfied between g and h we make some remarks on the problem of the quantum group dyamics.

Take the $\mathrm{SU}(2)_q$ generators [10, 11]

$$J_{\pm} = \sum_{m=-j}^{j} \sqrt{[j \mp m][j \pm m + 1]}\, |jm \pm 1\rangle\langle jm| \tag{1}$$

and

$$J_3 = \sum_{m=-j}^{j} m|jm\rangle\langle jm| \tag{2}$$

where we use the symbol: $[x] = (q^x - q^{-x})/(q - q^{-1})$, where $q = e^h$ is the deformation parameter. From these generators we obtain the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \tag{3}$$

$$[J_+, J_-] = [2J_3] \tag{4}$$

and the Casimir operator

$$C = [J_3 + \tfrac{1}{2}]^2 + J_- J_+ = [J_3 - \tfrac{1}{2}]^2 + J_+ J_- . \tag{5}$$

The polar decomposition for operators is the analogue of the complex number decomposition, $z = r\,e^{i\phi}$, and is defined as $O = UH$ where U is a partial isometry (one-side-unitary) operator and H a Hermitian operator [12]. For the present case of the $SU(2)_q$ generators we will require, on physical grounds, the U operator to be unitary. First we check that

$$J_+ J_- = \sum_{m=-j}^{j} [j+m][j-m+1]|jm\rangle\langle jm| \tag{6}$$

$$J_- J_+ = \sum_{m=-j}^{j} [j-m][j+m+1]|jm\rangle\langle jm| \tag{7}$$

and then write the quantum polar decomposition [13]

$$J_+ = \sqrt{J_+ J_-}\, h^{-1} = h^{-1}\sqrt{J_- J_+} \tag{8}$$

$$J_- = h\sqrt{J_- J_+} = \sqrt{J_+ J_-}\, h. \tag{9}$$

When we fix the order of the basis elements as $\{|j, -j\rangle, |j, -j+1\rangle, \ldots, |jj\rangle\}$, we obtain the following matrix representation for h:

$$h = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ e^{i\phi R} & & & & & 0 \end{bmatrix} \tag{10}$$

which is a $(2j+1)$-dimensional unitary matrix which obeys the relation $hh^{-1} = h^{-1}h = \mathbf{1}$, where $h^{-1} = h^+$ and the exponent $e^{i\phi_R}$, where ϕ_R is a reference phase, manifest the fact that the polar decomposition above is unique up to this phase choice. Hereafter we choose $\phi_R = 0$. We now observe that the matrix h together with the matrix

$$g = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{2j} \end{bmatrix} \tag{11}$$

where $\omega = e^{i2\pi/(2j+1)}$, i.e. the $2j+1$ primary root of unity, have the following commutator:

$$\omega gh - hg = 0 \tag{12}$$

and can be considered as the components (g, h) of a quantum plane [14] for the specific value of some deformation parameter $q_0 \equiv \omega = e^{i2\pi/(2j+1)}$. Several interesting things can now be noted. First, in the classical limit $q \to 1$, $\hbar \to 0$ the LHS of (8) and (9) becomes the classical $J_\pm$ generators while in the RHS the same is true for the operators under the square root but the operators h, h^{-1} remain the same since their matrix elements are not deformed. So the h, h^{-1} operators are the same in the quantum and classical polar decomposition of the algebra. Second, the (g, h) as matrix representations of elements of a quantum plane are related to the $GL(2)_{q_0}$ since any matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of this group has by definition the property of preserving the deformed 'scalar' product in the quantum plane [14, 15], namely,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} g' \\ h' \end{pmatrix} \tag{13a}$$

and

$$\omega g h - h g = \omega g' h' - h' g = 0 \tag{13b}$$

as long as a, b, c and d obey the relations:

$$\begin{aligned} ab &= \omega^{-1} ba \qquad & cd &= \omega dc \\ ac &= \omega^{-1} ca \qquad & bc &= cd \\ bd &= \omega^{-1} db \qquad & ad - da &= (\omega^{-1} - \omega) bc \end{aligned} \tag{14a}$$

while a, b, c, d commute with g, h and the quantum determinant, defined as

$$\det_{q_0=\omega} M = ad - \omega^{-1} bc \neq 0 \tag{14b}$$

commutes with all the elements. In the present case h has a direct physical meaning since in the polar decomposition in (8) and (9) it describes the exponential phase operator $e^{i\Phi}$, also $g = e^{\ln\omega(J_3 + j1)}$. As we have already pointed out (after (12)) in the classical limit of $SU(2)_q$, $q \to 1$, h as well as g remains the same†. Thus it appears that even after taking the classical limit of $SU(2)_q$ in the polar decomposition there is something remaining which is intrinsically quantum, namely the unitary phase operators h, h^{-1}.

We now elaborate on the contraction limit of the $SU(2)_q$ to the quantum oscillator algebra [6]. In the contraction limit $j \to \infty$ and $q > 1$, $q^j \to \infty$ while $[j] \equiv (q^j - q^{-j})/(q - q^{-1}) \to q^j (q - q^{-1})$. Defining the operators $h_\pm \equiv J_\pm / \sqrt{[2j]}$ and $h_3 \equiv J_3 + j\mathbf{1}$, we obtain the commutation relations

$$[h_+, h_-] = \frac{[2h_3 - 2j\mathbf{1}]}{[2j]} \tag{15}$$

and

$$[h_3, h_\pm] = \pm h_\pm. \tag{16}$$

The contracted matrix form of the h_- generators, for example, reads

$$h_- = \begin{bmatrix} 0 & \sqrt{[1]} & & \\ & 0 & q^{-1/2}\sqrt{[2]} & \\ & & 0 & q^{-1}\sqrt{[3]} \\ & & & \ddots \end{bmatrix} = q^{-h_3/2} a(q) = q^{1/2} a(q) q^{-h_3/2} \tag{17}$$

† We recall that in (8), (9) in the LHS we have quantum generators of $SU(2)_q$, for *any* q, while in the RHS we have h as a component in the quantum plane [14] which together with g, the second component, obeys the quantum plane condition (12) for the *specific* value of $q_0 = \omega = e^{i2\pi/(2j+1)}$.

where the deformed annihilation operator $a(q)$ occurring in the contraction of h_- is given by

$$a(q)=\sum_{n=0}^{\infty}\sqrt{[n]}|n-1\rangle_{qq}\langle n|. \tag{18}$$

Then the contraction of the commutator in (15) yields,

$$qq^{-h_3/2}a^{\dagger}aq^{-h_3/2}-q^{-h_3/2}aa^{\dagger}q^{-h_3/2}=-q^{-2h_3}$$

which can be written as [10]

$$aa^{\dagger}-qa^{\dagger}a=q^{-N} \tag{19}$$

where we have denoted the h_3 operator in the contraction limit, by the number operator

$$N=\sum_{n=0}^{\infty}n|n\rangle_{qq}\langle n|. \tag{20}$$

Similarly, we derive from (16) the commutators

$$[N,a^{\dagger}]=a^{\dagger} \qquad [N,a]=-a.$$

For $q<1$ we replace in the above contractions q by q^{-1}. The quantum polar decomposition when contracted will give

$$a=\sqrt{aa^{\dagger}}h=h\sqrt{a^{\dagger}a}$$
$$a^{\dagger}=h^{-1}\sqrt{a^{\dagger}a}=\sqrt{aa^{\dagger}}\,h^{-1}.$$

This is the polar decomposition for the q-bosonic operators containing in the square root the product of the q-creation and q-annihilation operators which is not however the number operator.

The relationship between the quantum $SU(2)_q$ generators with h, occurring in the polar decomposition, and its conjugate g motivates the embedding of the matrix realisation of the $SU(2)_q$ algebra into the $SU(2j+1)$ algebra formed by combinations of monomials in g and h. Let us consider j as an integer odd number (for even values of j the proof is similar). Then according to [16] we have

$$|k\rangle\langle k|=\frac{1}{2j+1}\sum_{m=1}^{2j+1}\omega^{(1-k)m}g^{m} \tag{21}$$

and

$$|k+1\rangle\langle k|=\frac{1}{2j+1}\sum_{m=1}^{2j+1}\omega^{-km}g^{m}\cdot h^{+1} \tag{22}$$

$$|k-1\rangle\langle k|=\frac{1}{2j+1}\sum_{m=1}^{2j+1}\omega^{-(k-2)m}g^{m}\cdot h. \tag{23}$$

The operators defined as

$$J_{\boldsymbol{m}}=J_{m_1m_2}=\omega^{m_1m_2/2}g^{m_1}h^{m_2} \tag{24}$$

provide a representation of the $SU(2j+1)$ classical algebra [17] with commutation relations

$$[J_{\boldsymbol{m}},J_{\boldsymbol{n}}]=2\mathrm{i}\sin\left(2\pi\frac{\boldsymbol{n}\times\boldsymbol{m}}{2j+1}\right)J_{\boldsymbol{m}+\boldsymbol{n}} \tag{25}$$

mod$(2j+1)$. We combine (21)-(23) and (1), (2) to embed the matrix representation of the $SU(2)_q$ quantum generators into the $SU(2j+1)$ classical algebra with deformed coefficients of expansion as follows:

$$J_+=\frac{1}{2j+1}\sum_{k=-j}^{j}\sum_{m=-j}^{j}\sqrt{[j-k][j+k+1]}\,\omega^{-k(m+j+1)}g^{m+j+1}h^{-1} \tag{26}$$

$$J_-=\frac{1}{2j+1}\sum_{k=-j}^{j}\sum_{m=-j}^{j}\sqrt{[j+k][j-k+1]}\,\omega^{-(k-2)(m+j+1)}g^{m+j+1}h \tag{27}$$

$$j_3=\frac{1}{2j+1}\sum_{k=-j}^{j}\sum_{m=-j}^{j}k\omega^{(m+j+1)(1-k)}g^{m+j+1} \tag{28}$$

or in terms of the J_m generators of the $SU(2j+1)$ algebra, as:

$$J_+=\frac{1}{2j+1}\sum_{k=-j}^{j}\sum_{m=-j}^{j}\sqrt{[j-k][j+k+1]}\,\omega^{-(k-1/2)(m+j+1)}J_{m+j+1,-1} \tag{29}$$

$$J_-=\frac{1}{2j+1}\sum_{k=-j}^{j}\sum_{m=-j}^{j}\sqrt{[j+k][j+k+1]}\,\omega^{-(k-3/2)(m+j+1)}J_{m+j+1,1} \tag{30}$$

$$J_3=\frac{1}{2j+1}\sum_{k=-j}^{j}\sum_{m=-j}^{j}k\omega^{(m+j+1)(1-k)}J_{m+j+1,0}. \tag{31}$$

The above embedding in the classical limit $q\to 1$, $[x]\to x$, goes over to the classical embedding of the SU(2) algebra into $SU(2j+1)$. We would like to emphasise that the common part between the SU(2) classical and quantum algebras appears to be, besides J_3, the 'exponential phase operator' h (and its conjugate g) which remain undeformed during quantisation.

We make a further remark on quantum dynamics. A Hamiltonian function $H(g, h)$ of g, h which obey the quantum plane condition, (12) acts as an endomorphism for the quantum plane in time. It is then tempting to say that the evolution operator of this Hamiltonian belongs to the $GL(2)_{q_0=\omega}$ group, and this group acts as a dynamic group. However this is not the case since the elements of such a matrix M which obey (14) should commute with g and h [14], while here the Hamiltonian depends on g, h and is not in general commuting with them. Representations of the $GL(2)_{q_0}$ matrix in terms of g, h have been given [18]; their elements obey (14) by construction but acting on the g, h doublet fail to preserve the quantum plane condition.

Finally, we briefly discuss the problem of the deformation of the Heisenberg equation of motion. To this end, we rewrite the deformed commutator (19) as

$$bb^{\dagger}-q^2b^{\dagger}b=1 \tag{32}$$

where $b\equiv q^{N/2}a$ and $b^{\dagger}\equiv a^{\dagger}q^{N/2}$. As a first alternative one could replace the ordinary commutator in the Heisenberg equation by a q-commutator. But in this case the Hamiltonian does not q-commute with itself, so is not a constant of motion. Thus this alternative is easily rejected.

In a more general case, one can replace the usual Heisenberg equation of motion, $\mathrm{i}\dot{f}=[f, H]$, by

$$\mathrm{i}D_t f=[f, H]_r \tag{33}$$

in which the usual time-derivative is replaced by the deformed time-derivative†,

$$D_t f(t) \equiv \frac{f(p^{1/2}t) - f(p^{-1/2}t)}{p^{1/2}t - p^{-1/2}t}$$

with an arbitrary deformation parameter p, and the deformed commutator

$$[f, H]_r \equiv fH - rHf$$

with arbitrary deformation parameter r, has been used.

One can check that the requirement of invariance of the q-commutation relation (32), i.e.

$$D_t(bb^\dagger - q^2 b^\dagger b) = 0$$

using (33) for $f = b$ and $b^\dagger$, is fulfilled only for the values $p = r = 1$, which brings us back to the original Heisenberg equation of motion.

It is interesting to study the question of the deformation of the dynamics further.

We are especially grateful to P Kulish for several useful discussions and clarifications. We thank J Lukierski for reading the manuscript and for helpful comments.

References

[1] Faddeev L D 1984 *Les Houches Lectures 1982* (Amsterdam: Elsevier)
Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* vol 151 (Berlin: Springer) p 61
[2] Kulish P P and Reshetikhin N Yu 1981 *Zap. Nauch. Semin. LOMI* **101** 101 (in Russian); 1983 *J. Sov. Math.* **23** 2435
Sklyanin E K 1982 *Func. Anal. Appl.* **16** 27; 1983 *Func. Anal. Appl.* **17** 34
[3] Drinfeld V G 1986 *Proc. Int. Congr. Math. Berkeley* **1** 798
[4] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[5] Kulish P P 1988 *Preprint* RIMS-615 Kyoto; 1989 *Proc. Alushta School on QFTh* (Berlin: Springer)
Kulish P P and Reshetikhin N 1989 *Lett. Math. Phys.* **18** 143
[6] Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
[7] Dirac P A M 1927 *Proc. R. Soc.* A **111** 243
[8] Carruthers P and Nieto M M 1968 *Rev. Mod. Phys.* **40** 411
[9] Pegg D T and Barnett S M 1988 *Europhys. Lett.* **6** 483
Barnett S M ahd Pegg D T 1989 *J. Mod. Opt.* **36** 7
Popov V N and Jarunin V S 1973 *Bull. Leningrad Univ.* **22** 7
[10] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[11] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[12] Reed M and Simon B 1972 *Methods in Modern Mathematical Physics I. Functional Analysis* (New York: Academic)
[13] Ellinas D 1990 Phase operators via group contraction *preprint*
[14] Manin Yu I 1988 *Quantum Groups and Non-Commutative Geometry* Centre de Recherches Mathématiques, University of Montreal
Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613
[15] Corrigan E, Fairlie D B, Fletscher P and Sasaki R *Preprint* University of Durham DTP-89/29
[16] Patera J and Zassenhaus H 1988 *J. Math. Phys.* **29** 665
[17] Hopp J 1987 *Workshop on Contraint's Theory and Relativistic Dynamics* ed G Longhi and L Lusann (Singapore: World Scientific)
Floratos E G and Iliopoulos J 1988 *Phys. Lett.* **201B** 237
Arakelyan T and Savvidy G K 1988 *Phys. Lett.* **214B** 350
Fairlie D, Fletcher P and Zachos C 1989 *Phys. Lett.* **218B** 203
Fairlie D and Zachos C 1989 *Phys. Lett.* **224B** 101
Fairlie D, Fletcher P and Zachos C *Preprint* University of Durham DTP-89/37
[18] Floratos E G *Preprint* CERN-Th 5482/89
[19] Ruegg H *Preprint* Universite de Geneve UGVA-DPT 1989/08-625

† For the use of q-derivatives to obtain the representations of $SU(2)q$, compare with [19].